Average entropy of a subsystem from its average Tsallis entropy

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(Received 8 August 2001; published 5 April 2002)

In the nonextensive Tsallis scenario, Page's conjecture for the average entropy of a subsystem [Phys. Rev. Lett. **71**, 1291 (1993)] as well as its demonstration are generalized, i.e., when a pure quantum system, whose Hilbert space dimension is mn, is considered, the average Tsallis entropy of an m-dimensional subsystem is obtained. This demonstration is expected to be useful to study systems where the usual entropy does not give satisfactory results.

DOI: 10.1103/PhysRevE.65.046131 PACS number(s): 05.70.Ln, 03.65.−w, 05.30.Ch, 05.90.+m

I. INTRODUCTION

Entropy is one of the most ubiquitous quantities in physics. For example, the entropy is fundamental in the study of the quantum- and classical-information theories, applied in recent developments in telecommunications, computer science, and engineering (for a review, see [1,2]). In particular, a great effort has been made to understand the quantum entanglement of inseparable quantum systems [3,4]. A traditional example of a pure entangled state is the Einstein-Podolsky-Rosen singlet state [5]. Another interesting aspect is to obtain information about the entropy of a subsystem by studying its average [6,7] over pure states of the big system, in unitary Haar measure. For instance, a complete pure system can be identified with a black hole and the radiation field related to it, in which case the subsystem is the black hole or alternatively the radiation field [8].

The standard entropy and its corresponding thermostatistics present serious difficulties when employed to study systems with a long-range interaction, in particular, when we deal with gravitational interactions [9–13]. A possible way to overcome this kind of difficulty is considering a new entropy. As stressed by Lavenda and co-workers, a newly proposed entropy should have concavity property [14]. Such an entropy was considered by Tsallis [15].

The Tsallis entropy

$$S^{(q)}(p_i)=\frac{1-\sum_i p_i^q}{q-1} \tag{1}$$

recovers the usual entropy $S(p_i)=S^{(1)}(p_i)=-\Sigma_i p_i \ln p_i$ in the limit $q\to1$ and has a definite concavity for all q values ($S^{(q)}$ is concave for $q>0$ and convex for $q<0$). Furthermore, if we consider two independent subsystems A and B, we have the probabilities $p_{ij}^{AB}=p_i^A p_j^B$, and

$$S_{AB}^{(q)}=S_A^{(q)}+S_B^{(q)}+(1-q)S_A^{(q)}S_B^{(q)}, \tag{2}$$

in contrast with the extensive property of the usual entropy, $S_{AB}=S_A+S_B$. Thus, the parameter q gives a measure of the nonextensivity induced by the Tsallis entropy. In this context, it is common to employ the jargon "nonextensive" to refer to the scenario when the Tsallis entropy is present.

Many investigations based on the Tsallis entropy have been developed. A representative set of such developments relates to self-gravitating systems [16], cosmic background radiation [17], peculiar velocities in galaxies [18], Lévy-type anomalous superdiffusion [19], H theorem [20], turbulence [21], nonlinear anomalous diffusion [22], perturbation and variational methods [23], linear response theory [24], Green's functions [25], and quantum entanglement [26] (for a recent review see Ref. [27]).

Since the Tsallis entropy has played a central role in a nonextensive scenario, such as those cited previously, it is natural to investigate this generalized entropy further. A different reason for investigating the Tsallis entropy $S^{(q)}$ is to technically sneak up on ordinary entropy S, yet avoiding its annoying logarithm by exploiting the $q\to1$ limit. In any case, the aim of this work is to obtain the Tsallis entropy of a subsystem averaged over all pure states of the total system using unitary Haar measure to define our averaging. This result generalizes Page's conjecture [6] (a formula for that average of the usual entropy of a subsystem) and its subsequent demonstration [28,29]. We note that Page's conjecture for the average entropy of a subsystem has been applied to investigate black hole radiation [8]; perhaps our generalization can be useful to study parallel reductions to a subsystem in attempts to fit data with a Tsallis q distinct from 1.

To present our generalization, it is useful to first review Page's work. This is performed in Sec. II. Section III is addressed to calculate the average Tsallis entropy of a subsystem. A summary is given in the last section.

II. AVERAGE ENTROPY OF A SUBSYSTEM

One way to get entropy out of a system in a pure quantum state is by a coarse graining of dividing the system into two subsystems and ignoring their correlations. Take the system AB with Hilbert space dimension mn and normalized density matrix ρ_{AB} and divide it into two subsystems A and B, of dimensions m and n, respectively. The entropy of the system A is $S_A=-\mathrm{tr}\rho_A \ln\rho_A$, where the density matrix of the system A is obtained by taking a partial trace over a total system, $\rho_A=\mathrm{tr}_B\rho_{AB}$. In the same way, $S_B=-\mathrm{tr}\rho_B\ln\rho_B$, with $\rho_B=\mathrm{tr}_A\rho_{AB}$. If the system AB is in a pure state, then $S_{AB}=0$ and $S_A=S_B$ as a consequence of the fact that ρ_A and ρ_B have the same set of nonzero eigenvalues [30]. Unless the two systems are uncorrelated in the quantum sense ($\rho_{AB}=\rho_A \otimes\rho_B$, in which case $S_A=S_B=0$), a full quantum analysis is necessary in order to obtain S_A and S_B, which can be cumbersome. Yet it is sometimes easy to calculate the unitary Haar average entropy of the subsystem A over all pure states of the total system, $S_{m,n}=\langle S_A\rangle$, and consequently also the average information of the subsystem, i.e., the deficit of the average entropy from the maximum, $I_{m,n}=S^m_{\max}-\langle S_A\rangle$, with $S^m_{\max}=S(p_i=1/m)$.

For $m\leqslant n$, Page showed that

$$S_{m,n}=\int S(p_i)P(p_1,\ldots,p_m)dp_1\ldots dp_m, \tag{3}$$

where $S(p_i)=-\Sigma_{i=1}^m p_i\ln p_i$, and $P(p_1,\ldots,p_m)$ is the probability distribution of the eigenvalues of ρ_A for the random pure states ρ_{AB} of the entire system [6,7],

$$P(p_1,\ldots,p_m)dp_1\ldots dp_m=N\delta\left(1-\sum_{t=1}^m p_l\right)\prod_{1\leqslant i<j\leqslant m}(p_i-p_j)^2\prod_{k=1}^m p_k^{n-m}dp_k. \tag{4}$$

In Eq. (3), as well as in the following integrals, the integration limits are 0 and ∞. In the above equation, $N=1/\int P(p_1,\ldots,p_m)dp_1\ldots dp_m$ is the normalization constant.

By using the identity $1=(\int r^{nm}e^{-r}dr)/(mn\int r^{nm-1}e^{-r}dr)$ and the polygamma function $\Psi(mn+1)=(\int\ln r r^{nm}e^{-r}dr)/(mn\int r^{nm-1}e^{-r}dr)$, we can write Eq. (3) as

$$S_{m,n}=-\frac{\int e^{-r}r^{mn}\sum_i p_i\ln p_i P(p_1,\ldots,p_m)dp_1\ldots dp_m dr}{mn\int e^{-r}r^{mn-1}P(p_1,\ldots,p_m)dp_1\ldots dp_m dr}-\frac{\int \ln r e^{-r}r^{mn}P(p_1,\ldots,p_m)dp_1\ldots dp_m dr}{mn\int_0^\infty e^{-r}r^{mn-1}P(p_1,\ldots,p_m)dp_1\ldots dp_m dr}+\Psi(mn+1). \tag{5}$$

Taking into account that $\Sigma_i p_i=1$, we can introduce the new variables $x_i=rp_i$; then, by using the δ function to evaluate the integral in r, we obtain

$$S_{m,n}=\Psi(mn+1)-\frac{\int S(x_i)Q(x_1,\ldots,x_m)dx_1\ldots dx_m}{mn\int Q(x_1,\ldots,x_m)dx_1\ldots dx_m}, \tag{6}$$

with

$$Q(x_1,\ldots,x_m)dx_1\ldots dx_m=\prod_{1\leqslant i<j\leqslant m}(x_i-x_j)^2\times\prod_{k=1}^m e^{-x_k}x_k^{n-m}dx_k. \tag{7}$$

Page conjectured [6], and other authors proved [28,29], that the exact result is

$$S_{m,n}=\sum_{k=n+1}^{mn}\frac{1}{k}-\frac{m-1}{2n}. \tag{8}$$

Page had meanwhile applied this to calculate the information in black hole radiation [8]. It was considered a pure composite total state with a fixed dimension *mn*, composed of the black hole and the radiation. The radiation subsystem has dimension *m* and the black hole one has dimension *n*. The average information in the smaller subsystem (for example, if you have $1\ll m\leqslant n$) is $I_r=S^m_{\max}-\langle S_r\rangle\approx m/2n$. If furthermore $m\ll n$, the smaller subsystem is very nearly maximally mixed, and has very little information in it. The information increases for higher dimension of the smaller subsystem.

III. AVERAGE TSALLIS ENTROPY

In this work, the above result is generalized to "the nonextensive case" as defined by replacing the usual entropy $[S(p_i)]$ in Eq. (3) by the Tsallis entropy $[S^{(q)}(p_i)]$. After similarly introducing the variables $x_i=rp_i$ in this generalization of Eq. (3), we obtain

$$S^{(q)}_{m,n}=\frac{1}{q-1}-\frac{1}{q-1}\frac{\Gamma(mn)}{\Gamma(mn+q)}J^{(q)}_{m,n}, \tag{9}$$

where

$$J^{(q)}_{m,n}=\frac{\int\sum_{i=1}^m x_i^q Q(x_1,\ldots,x_m)dx_1\ldots dx_m}{\int Q(x_1,\ldots,x_m)dx_1\ldots dx_m}. \tag{10}$$

This expression can be written as a one-dimensional integral in terms of the one-point correlation function of a Laguerre ensemble of complex Hermitian random matrices [31]. By considering the symmetry of x_i and the van der Monde determinant $\Delta_m(x)=\Pi_{1\leqslant i<j\leqslant m}(x_i-x_j)$, Eq. (10) reduces to

$$J^{(q)}_{m,n}=\int dx_1 x_1^q\chi(x_1), \tag{11}$$

where

$$\chi(x_1)=\frac{m\int|\Delta_m(x)|^2\prod_{k=1}^m\mu(x_k)dx_2\ldots dx_m}{\int|\Delta_m(x)|^2\prod_{k=1}^m\mu(x_k)dx_1\ldots dx_m} \tag{12}$$

with a weight function $\mu(x)=x^{n-m}e^{-x}$. This integration gives

$$\chi(x_1)=\frac{m!}{(n-1)!}x_1^{n-m}e^{-x}\{[L_{m-1}^{n-m+1}(x_1)]^2 - L_{m-2}^{n-m+1}(x_1)L_m^{n-m+1}(x_1)\}, \tag{13}$$

where $L_r^{\alpha}(x)$ are the associated Laguerre polynomials [31] (see also Ref. [28]).

The remaining integration in $J_{m,n}^{(q)}$, Eq. (11), can be evaluated by taking the following result [32]:

$$\int_0^{\infty} x^{\theta}e^{-x}L_r^{\alpha}(x)L_s^{\beta}(x)dx=\sum_{k=0}^{\min(r,s)}(-1)^{r+s}\binom{\theta-\alpha}{r-k}\binom{\theta-\beta}{s-k}\frac{\Gamma(\theta+k+1)}{k!}, \tag{14}$$

where $\theta>-1$, α and β are real parameters; and the brackets are binomial coefficients whose factorials of nonintegers or integers $\leqslant 0$ are interpreted through the usual $z!=\Gamma(z+1)$.

We finally get to our goal, a computationally explicit generalization of Page's conjecture as well as its demonstration, i.e.,

$$S_{m,n}^{(q)}=\frac{1}{q-1}-\frac{1}{q-1}\frac{\Gamma(m+1)\Gamma(mn)}{\Gamma(n)\Gamma(mn+q)}\times\left[\sum_{k=0}^{m-1}\binom{q-1}{m-1-k}^2\frac{\Gamma(n-m+q+1+k)}{k!}-\sum_{k=0}^{m-2}\binom{q-1}{m-2-k}\binom{q-1}{m-k}\frac{\Gamma(n-m+q+1+k)}{k!}\right] \tag{15}$$

for $m\leqslant n$.

In the following, we discuss $S_{mn}^{(q)}$, mainly its dependence on q. Note that Page's result, Eq. (8), is recovered from $S_{mn}^{(q)}$ by taking the appropriate limit $(q\rightarrow 1)$, i.e., in this limit, Eq. (15) reduces to

$$\begin{aligned}S_{m,n}^{(q\rightarrow 1)}={}&\Psi(mn+1)-\frac{\Gamma(m+1)\Gamma(mn)}{\Gamma(n)\Gamma(mn+1)}\left[\sum_{k=0}^{m-1}\frac{\Gamma(n-m+2+k)}{[\Gamma(m-k)\Gamma(k-m+2)]^2k!}\right.\\&\left.\times[2\Psi(1)-2\Psi(k-m+2)+\Psi(n-m+2+k)]\right]\\&+\frac{\Gamma(m+1)\Gamma(mn)}{\Gamma(n)\Gamma(mn+1)}\left[\sum_{k=0}^{m-2}\frac{\Gamma(n-m+2+k)}{\Gamma(m-k+1)\Gamma(k-m+1)\Gamma(m-k-1)\Gamma(k-m+3)k!}\right.\\&\left.\times\{2\Psi(1)-\Psi(k-m+1)-\Psi(3-m+k)+\Psi(n-m+2+k)\}\right].\end{aligned} \tag{16}$$

In the above equation, the only nonvanishing term in the summation is that one corresponding to k maximum, so that we obtain $S_{m,n}^{(q\rightarrow 1)}=\Psi(nm+1)-\Psi(n+1)-(m-1)/2n$. By using the relation $\Psi(n+1)=\Sigma_{k=0}^{n}1/k-\gamma$, where γ is the Euler's constant, we get Page's results, Eq. (8).

Furthermore, as in the case $q=1$, $S_{mn}^{(q)}$ also assumes a simple form when q is a positive integer. This is a consequence of poles of the $\Gamma(x)$ function for negative integers x. Thus, in the cases of $q=2,3,4,\ldots$, Eq. (15) reduces to

$$\begin{aligned}S_{m,n}^{(q)}={}&\frac{1}{q-1}-\frac{1}{q-1}\frac{\Gamma(m+1)\Gamma(mn)}{\Gamma(n)\Gamma(mn+q)}\\&\times\left[\sum_{k=1}^{q}\left(\frac{\Gamma(q)}{\Gamma(k)\Gamma(q+1-k)}\right)^2\frac{\Gamma(n+q+1-k)}{(m-k)!}\right.\\&-\sum_{k=1}^{q-2}\left(\frac{\Gamma(q)}{\Gamma(k)\Gamma(q+1-k)}\right)\\&\left.\times\left(\frac{\Gamma(q)}{\Gamma(2+k)\Gamma(q-1-k)}\right)\frac{\Gamma(n+q-k)}{(m-1-k)!}\right].\end{aligned} \tag{17}$$

Note that the second sum only gives contribution for $q=3,4,5,\ldots$. In particular, for $q=2$, the Tsallis entropy leads to the quadratic entropy. This entropy was first used in theoretical physics by Fermi (see Ref. [33]). In this case, Eq. (17) reduces to [34]

$$S_{m,n}^{(q=2)}=1-\frac{n+m}{mn+1}. \tag{18}$$

If we observe that the maximum q entropy, obtained when $p_i=1/m$, is given by $S_{\max}^{(q)m}=(1-m^{1-q})/(q-1)$, the average information, $I_{m,n}^{(q)}=S_{\max}^{(q)m}-\langle S_A^{(q)}\rangle$, for $q=2$ is

$$I_{m,n}^{(q=2)}=\left(1-\frac{1}{m}\right)-\left(1-\frac{m+n}{mn+1}\right)\approx\frac{1}{n}, \tag{19}$$

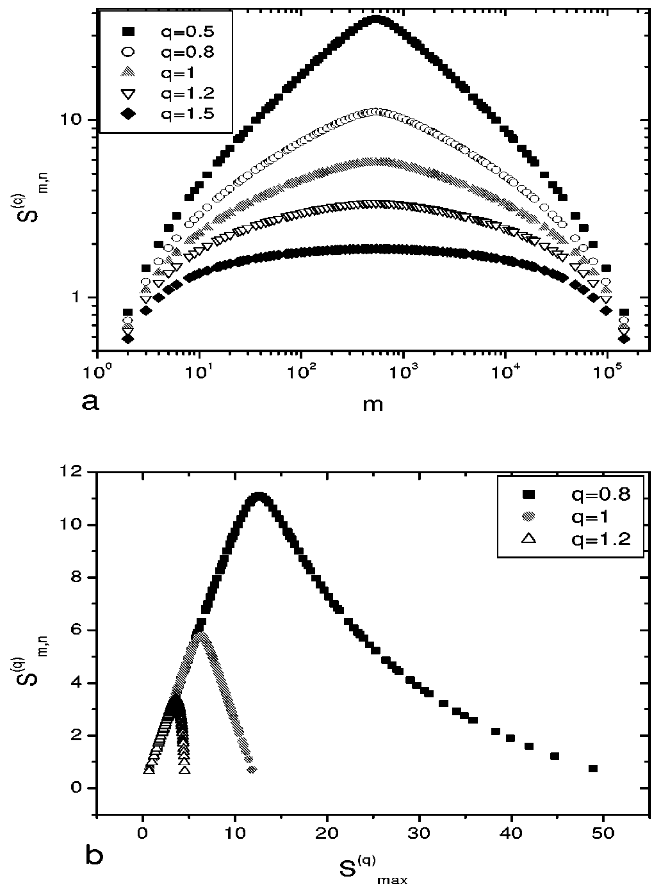

FIG. 1. (a) Plot of $\langle S_A^{(q)}\rangle$ versus m to $q=0.5$, $q=0.8$, $q=1$, $q=1.2$, and $q=1.5$ with $mn=291\,600$. (b). Plot of $\langle S_A^{(q)}\rangle$ versus $S_{\max}^{(q)}$ to $q=0.8$, $q=1$ and $q=1.2$ with $mn=291\,600$.

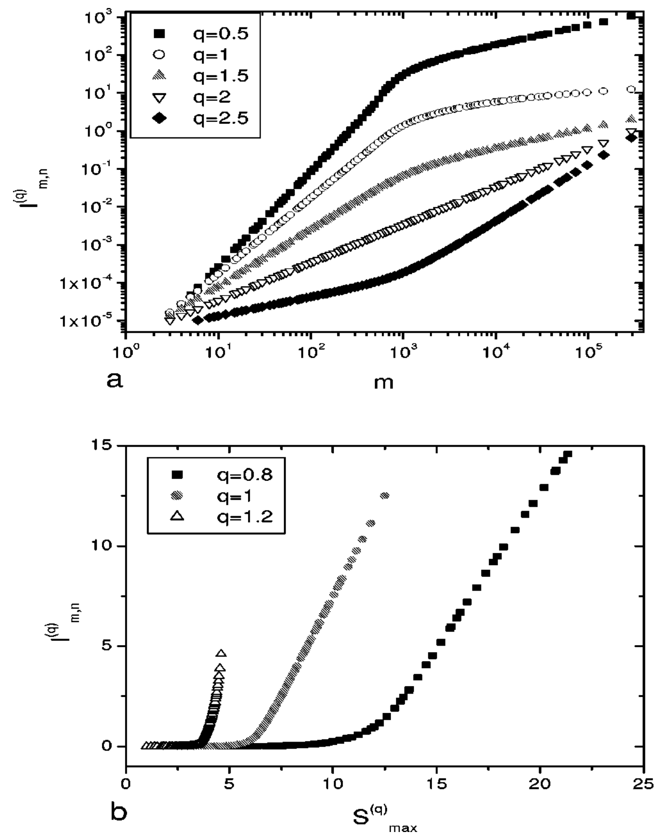

FIG. 2. (a) Plot of $I_{m,n}^{(q)}$ versus m to $q=0.5$, $q=1$, $q=1.5$, $q=2$, and $q=2.5$ with $mn=291\,600$. (b) Plot of $I_{m,n}^{(q)}$ versus $S_{\max}^{(q)}$ to $q=0.8$, $q=1$, and $q=1.2$ with $mn=291\,600$.

for $mn \gg 1$. Observe that for $mn \gg 1$, $I_{m,n}^{(q=2)}$ is a power law with only n dependence. Thus, for a system AB with fixed mn dimension, a log-log plot of $I_{m,n}^{(q=2)}$ versus m gives a straight line.

For an arbitrary q value, Eq. (15) does not reduce to a simple form, so we show some graphs instead. For example, consider a total system with fixed Hilbert space dimension $mn=291\,600$ (about the number of states very naively expected for a black hole near the Planck mass [8]). In the case of a total pure state, we have $\langle S_A^{(q)}\rangle = \langle S_B^{(q)}\rangle = S_{m,n}^{(q)}$ if $m \leqslant n$, and $\langle S_A^{(q)}\rangle = \langle S_B^{(q)}\rangle = S_{n,m}^{(q)}$ if $m \geqslant n$, where $S_{m,n}^{(q)}$ is given by Eq. (15) and $S_{n,m}^{(q)}$ is obtained from it by performing the exchange $m \leftrightarrow n$. In Fig. 1, we plot $\langle S_A^{(q)}\rangle$ for some representative q values. Figure 2 shows the average information $I_{m,n}^{(q)}$ to different q values.

IV. SUMMARY

Summing up, we have generalized Page's conjecture and its demonstration in order to incorporate the nonextensive regime induced by the Tsallis entropy. Naturally, this result must and does reduce to the usual one in the limit $q \to 1$. For other representative q values and mn still fixed at 291 600, average entropy and average information are log-log plotted, $S^{(q)}$ versus m then $S^{(q)}$ versus $S_{\max}^{(q)}$ in Fig. 1, and $I^{(q)}$ versus m then $I^{(q)}$ versus $S_{\max}^{(q)}$ in Fig. 2. The straightness shown by the triangles in Fig. 2(a) illustrates the case $q=2$ as a separation between two different regimes. In general, calculations based on the nonextensive Tsallis entropy have been addressed in the study of systems with a long-range interaction, spatiotemporal complexity, and fractal structure; thus, we hope our result may be useful for such systems.

More formal applications of the $q \to 1$ limit to derive ordinary entropies, may also turn out feasible, for kinds of averaging other than Haar unitary, in particular, for time averaging under Gaussian-distributed Hamiltonians that do not discriminate between m system and n system, and also for similar distributions that, instead, do discriminate so as to model approximate mutual isolation. In both cases, results for $q=2$ are known [35] and the $q \to 1$ limit would be welcome. Such further applications would be analogous to our demonstration in this present paper of Page's conjecture, in being independent of the issue of whether the Tsallis entropy for $q \neq 1$ is or is not directly applicable to physical situations.

ACKNOWLEDGMENT

We thank CNPq (Brazilian Agency) for partial financial support.

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